

Closing Tue: Taylor Notes 1, 2
 Closing Thu: Taylor Notes 3
 Final: Sat, June 2, 5:00pm, KANE 130
 Eight pages, covers everything.

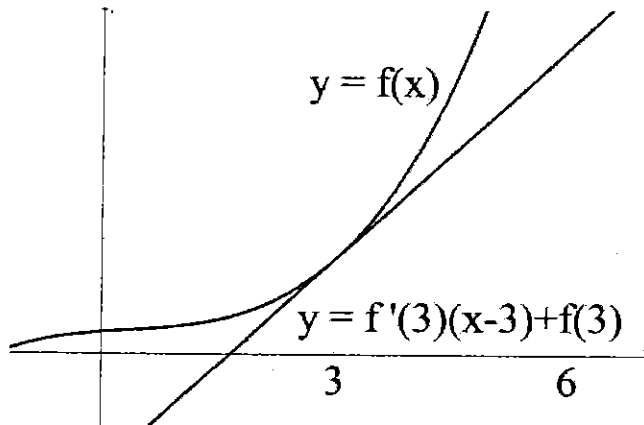
Taylor Notes 1 (TN 1):

Tangent Line Error Bounds

Goal: Approximate functions with tangent lines and get error bounds.

Def'n: The first Taylor polynomial for $f(x)$ based at b is

$$T_1(x) = f(b) + f'(b)(x - b)$$



Entry Task: Find the 1st Taylor polynomial for $f(x) = \sqrt{x}$ at $x = 4$.

- Use it to estimate $\sqrt{4.5}$.
- Use your calculator to find the difference between this estimate and the actual value.

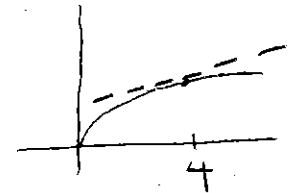
$$f(x) = \sqrt{x} \Rightarrow f(4) = \sqrt{4} = 2$$

$$f'(x) = \frac{1}{2\sqrt{x}} \Rightarrow f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$$

$$T_1(x) = 2 + \frac{1}{4}(x - 4)$$

For $x \approx 4$, we have

$$\sqrt{x} \approx 2 + \frac{1}{4}(x - 4)$$



Thus

$$\sqrt{4.5} \approx 2 + \frac{1}{4}(4.5 - 4) = 2.125$$

2.12182

$$\text{Error} = |f(4.5) - T_1(4.5)|$$

$$= |\sqrt{4.5} - 2.125|$$

$$= |-0.00367966| = 0.00367966$$

Correct to two digits!

Bounding the Error

Given an interval around $x = b$
(i.e. $b - a \leq x \leq b + a$).

Tangent Linear Error Bound Thm

If $|f''(x)| \leq M$ for all x , then

$$|f(x) - T_1(x)| \leq \frac{M}{2} |x - b|^2.$$

To use

Step 1: Find $f''(t)$.

Step 2: Find an upper bound (max)
for $|f''(t)|$ on the interval.
Put this in for M in thm.

Step 3: Plug in $x =$ "an endpoint" to
get *worst case* error bound.

Example: Using the theorem, give an
error bound for $|\sqrt{x} - T_1(x)|$ based
at $x = 4$ on the interval $[3.5, 4.5]$.

$$f'(x) = \frac{1}{2} x^{-1/2}$$

$$\Rightarrow f''(x) = -\frac{1}{4} x^{-3/2} = -\frac{1}{4x^{3/2}}$$

$$|f''(x)| = \frac{1}{4x^{3/2}} \quad \leftarrow \text{DECREASING FUNCTION, MAX MUST BE AT 3.5}$$

WHAT IS THE MAXIMUM
FOR $3.5 \leq x \leq 4.5$

$$|f''(x)| \leq \frac{1}{4(3.5)^{3/2}} = 0.03818 = M$$

$$|\sqrt{x} - T_1(x)| \leq \frac{0.03818}{2} |x - 4|^2$$

\uparrow
3.5 OR 4.5 { DOESN'T MATTER

$$\frac{0.03818}{2} (0.5)^2 = 0.0047725$$

ON THIS INTERVAL THE
ERROR IS NEVER BIGGER THAN
0.0047725

Example: $f(x) = \ln(x)$ at $b = 1$.

- Find the 1st Taylor polynomial.
- Use the error bound formula to find a bound on the error over the interval $J = [1/2, 3/2]$
- Find an interval around $b = 1$ where the error is less than 0.01.

(a) $f(x) = \ln(x) \Rightarrow f(1) = \ln(1) = 0$
 $f'(x) = \frac{1}{x} \Rightarrow f'(1) = \frac{1}{1} = 1$

$T_1(x) = 0 + 1(x-1) = x-1$
 $\ln(x) \approx x-1$ for $x \approx 1$

(b) $f''(x) = -\frac{1}{x^2}$
 $|f''(x)| = \frac{1}{x^2}$ ← DECREASING FUNCTION
 MAX WILL BE AT $\frac{1}{2}$

$\frac{1}{x^2} \leq \frac{1}{(\frac{1}{2})^2} = 4 = M$

 $|\ln(x) - (x-1)| \leq \frac{4}{2} |x-1|^2 \leq 2(0.5)^2 = 0.5$ ← EITHER ENDPOINT

x	f(x)	T ₁ (x)	f(x) - T ₁ (x)
1	0	0	0
1.2	0.1823	0.2	0.01768
1.4	0.3364	0.4	0.06353
0.9	-0.1053	-0.1	0.00536

(c) WANT $[1-a, 1+a]$
 SUCH THAT
 $|f(x) - T_1(x)| \leq \frac{M}{2} |x-1|^2 \leq 0.01$

 ON $[1-a, 1+a]$
 $|f''(x)| \leq \frac{1}{(1-a)^2} = M$
 $\frac{M}{2} |x-1|^2 = \frac{1}{2} \frac{1}{(1-a)^2} (1+a-1)^2$
 $= \frac{1}{2} \frac{a^2}{(1-a)^2} = 0.01$

$(\frac{a}{1-a})^2 = 0.02$
 $\frac{a}{1-a} = \sqrt{0.02}$
 $a = \sqrt{0.02} (1-a)$
 $(1 + \sqrt{0.02})a = \sqrt{0.02}$
 $a = \frac{\sqrt{0.02}}{1 + \sqrt{0.02}} \approx 0.1239$

$[1-0.1239, 1+0.1239]$

Proof of error bound for $x > b$:

Start with $f(x) - f(b) = \int_b^x f'(t) dt$.

Do integration by parts,

(with $u = f'(t)$, $dv = dt$,
 $du = f''(t)$, $v = t - x$)

to get

$$f(x) - f(b) = f'(b)(x - b) - \int_b^x (t - x)f''(t) dt$$

Rearrange to get

$$f(x) - f(b) - f'(b)(x - b) = \int_b^x (x - t)f''(t) dt$$

Thus,

$$|f(x) - T_1(x)| = \left| \int_b^x (x - t)f''(t) dt \right|$$

Then note

$$\begin{aligned} \left| \int_b^x (x - t)f''(t) dt \right| &\leq \int_b^x (x - t)|f''(t)| dt \\ &\leq M \int_b^x (x - t) dt \\ &\leq \frac{M}{2} (x - b)^2. \end{aligned}$$

Handwritten notes:
CAN BE ANY CONSTANT, WE CHOOSE x TO MAKE ANSWER COME OUT NICE
UPPER BOUND

Note about "Bounds":

An upper **bound**, M , is a number that is always bigger than the function.

The smallest possible upper bound is sometimes called a *tight* bound.

Examples: Find any upper **bound** (if it is easy to do so, find a *tight* upper bound).

1. $|\sin(5x)|$ on $[0, 2\pi]$

$$|\sin(5x)| \leq 1$$

↑
"TIGHT" UPPER
BOUND

~~1/2~~

2. $|x - 3|$ on $[1, 5]$

$$|x - 3| \leq 2 \quad \leftarrow \text{TIGHT}$$

3. $\left| \frac{1}{(2-x)^3} \right|$ on $[-1, 1]$

$$\left| \frac{1}{(2-x)^3} \right| \leq \frac{1}{(2-1)^3} = 1$$

WILL BE
LARGEST
WHEN
DENOMINATOR
IS SMALLEST
WHICH IS AT
 $x = 1$

4. $|\sin(x) + \cos(x)|$ on $[0, 2\pi]$
NOT TIGHT

$$|\sin(x) + \cos(x)| \leq 1 + 1 = 2 \quad \leftarrow$$

TIGHT UPPER BOUND

IS HARDER TO FIND THEN

$$|\sin(x) + \cos(x)| \leq \sqrt{2}$$

5. $|\cos(2x) + e^{2x} + \frac{6}{x}|$ on $[1, 4]$

$$|\cos(2x) + e^{2x} + \frac{6}{x}| \leq 1 + e^8 + 6 = 7 + e^8$$

NOT TIGHT

HARDER TO GET TIGHT UPPER BOUND

Example (you do):

Let $f(x) = x^{1/3}$ and $b = 8$.

- (a) Find the 1st Taylor Polynomial.
(b) Give a bound on the error over the interval $J = [7, 9]$.

$$f(x) = x^{1/3} \Rightarrow f(8) = 8^{1/3} = 2$$

$$f'(x) = \frac{1}{3} x^{-2/3} \Rightarrow f'(8) = \frac{1}{3} \frac{1}{8^{2/3}} = \frac{1}{3} \frac{1}{2^2} = \frac{1}{12}$$

$$T_1(x) = 2 + \frac{1}{12}(x-8)$$

$$x^{1/3} \approx 2 + \frac{1}{12}(x-8) \quad \text{for } x \approx 8$$

$$f''(x) = -\frac{2}{9} x^{-5/3} = -\frac{2}{9x^{5/3}} \quad \leftarrow \begin{array}{l} \text{DECREASING} \\ \text{FUNCTION} \end{array}$$

$$|f''(x)| \leq \frac{2}{9x^{5/3}} \leq \frac{2}{9(7)^{5/3}} \quad \begin{array}{l} \text{MAX AT} \\ x=7 \end{array}$$
$$\approx 0.008675 = M$$

$$|f(x) - T_1(x)| \leq \frac{M}{2} |x-8|^2$$

$$\frac{1}{2} (0.008675) |9-8|^2$$
$$= 0.0043375$$

Ex

$$\sqrt[3]{9} \approx 2 + \frac{1}{12}(9-8)$$
$$2 + \frac{1}{12} = \underline{2.08\bar{3}}$$

$$\text{ACTUAL} = \underline{2.0800838}$$